Singular elliptical operators

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In problems involving one (or more Black Holes (B.H.) when the excision technique is used, we can have to handle degenerate elliptical operators.

An example, is the equation for the shift β^i when the lapse N vanishes on the horizon. In fact the equation for the shift reads (in an apropiate gauge)

$$\nabla^j K_{ij} = 0 \tag{1}$$

where K^{ij} is the extrinsic curvature tensor

$$K_{ij} = \frac{1}{2N}(\nabla_i \beta_j + \nabla_j \beta_i - \partial_0 \gamma_{ij}) \tag{2}$$

Here in after we shall express all the differential operators in terms of the flat covariant derivative $\mathcal{D}_i$ computed with respect the flat metric f_{ik} that in spherical coordinates reads [1]

$$f_{11} = 1, \quad f_{22} = r^2, \quad f_{33} = r^2 \sin^2\theta \tag{3}$$

Under the hypothesis that the topology of the horizon is the topology of the sphere the equation of the horizon can be reduced to be

$$r = 1 \tag{4}$$

Consequently we have to solve the the Einstein equations in the *excised* space

$$1 \leq r \leq \infty \tag{5}$$

The technique used to solve the Einstein equations is to solve these equations in two domains

$$1 \leq r \leq 2, \qquad 2 \leq r \leq \infty \tag{6}$$

and to match the solutions and they first derivatives at $r = 2$
The shift equation(1) can be written

[1] See the paper by S.Bonazzola et al. *Phys.Rev.D* **70** (2004), 104007

$$\mathcal{D}_j\mathcal{D}^j\beta^i+\frac{1}{3}\mathcal{D}^i(\mathcal{D}^j\beta^j)-(\mathcal{D}^i\beta^j+\mathcal{D}^j\beta^i-\frac{2}{3}\mathcal{D}_l\beta^l f^{ij}+S_1^i)\frac{\partial_j N}{N}=S_2^i$$

With the B.C. $\beta^i = 0 \mid_{r=\infty}$. In order to match the solution and its derivative we must have at list one homogeneous solution in the domain $1 \leq r \leq 2$. Question : How many homogeneous solutions exist ?
Taking into account that near the horizon

$$N = (r-1)N_0(r,\theta,\phi)$$

we look for the homogeneous solutions of the equation

$$\mathcal{D}_j\mathcal{D}^j\beta^i-\frac{1}{3}\mathcal{D}^i\mathcal{D}_j\beta^i-\frac{1}{x}(\mathcal{D}^r\beta^i+\mathcal{D}^i\beta^r-\frac{2}{3}\mathcal{D}_l\beta^l f^{ir})=0$$

where

$$x = r-1$$

The vectorial operator of the above equation, in spherical coordinates and spherical components is quite messy. By introducing two angular potentials η and μ defined by the equations

$$\beta^\theta = \partial_\theta \eta - \frac{1}{\sin\theta}\partial_\phi \mu, \quad \beta^\phi = \partial_\theta \mu + \frac{1}{\sin\theta}\partial_\phi \eta$$

we have two coupled Poisson equations for β^r and η and a Poisson equation for μ that after an expansion in spherical harmonics reads:

$$\frac{d^2\mu}{dr^2}+\frac{2}{r}\frac{d\mu}{dr}-\frac{l(l+1)}{r^2}\mu-\frac{1}{x}(\frac{d\mu}{dr}-\frac{\mu}{r}) = 0, \quad (x = r-1)$$

A solution μ_1 can be found by making a power expansion

$$\mu_1 = x^2 - \frac{5}{3}x^3 + ...$$

For $l = 1$ it exists an other homogeneous solution:

$$\mu_2 = r$$

that means that a black hole can rigidly rotate. In fact, the non vanishing at $r = 1$ hmogenous solutions are

$$\mu_1 = r\cos\theta, \quad \mu_2 = r\sin\theta\cos\phi, \quad \mu_3 = r\sin\theta\sin\phi \tag{7}$$

from wich the corresponding solutions for β

$$\beta_r = 0, \quad \beta_\theta = 0, \quad \beta_\phi = r\sin\theta$$

$$\beta_r = 0, \quad \beta_\theta = r\sin\phi, \quad \beta_\phi = r\cos\theta\cos\phi$$

$$\beta_r = 0, \quad \beta_\theta = -r\cos\phi, \quad \beta_\phi = r\cos\theta\sin\phi$$

A similar analysis can be performed for the poloydal part β^r, η of th shift. The conclusions are:

For $l = 1$, two couples of homogeneous solutions exist. That means that a rigid translation of the horizon can be chosen.

For $l \neq 1$ only β^r can be given on the horizon: The horizon can breath.

Finally singular equations exist for the metric coefficients h^{ik}. For some coefficient (h^{rr}) a boundary condition on the horizon can be given, for other coefficients ($h^{\theta\theta}$) not.

Equation $\triangle G + \frac{1}{r-1}(k_1\frac{d}{dr} + \frac{k_2}{r})G = 0$

Consider the equation

$$\frac{d^2G}{dr^2}+k_0\frac{1}{r}\frac{dG}{dr}+\frac{1}{r^2}(-l(l+1)+k_l)G+\frac{1}{r-1}(k_1\frac{d}{dr}+\frac{k_2}{r})G = 0 \tag{8}$$

For $k_1 = k_2 = 0$ the above equation has two regular solution regular a t $r = 0$ and at $r = \infty$,k_l if $k_l = ((k_0 - 1)^2 - 1)/4$.

$$g_1 = r^j, j_1 = \frac{1 - k_0 - (2l + 1)}{2}, j_2 = \frac{1 - k_0 + (2l + 1)}{2}$$

Note that the two solutions r^{j_1} and r^{j_2} are integer numbers if k_0 is integer to. In this section, we study the number and the aanalytical properties of the solution for different values of the parameter k_0, k_1, k_2.

Case $k_1 \neq 0$ and $k_1 \neq \mid 1 \mid$

Without losses of generality we consider only the case $k_2 = 0$. In fact by putting $\bar{G} = Gr^{k_2/k1}$

the equation for the new function $\bar{G}$ will be transformed in an equation having $k_2 = 0$
The case $\mid k_1 \mid = 1$ was already discussed.

The technique used consists in studying the behavior of the solution around the singular point $r = 1$. For that we introduce the new variable $x = r - 1$. The Eq. 8 writes

$$\frac{d^2G}{dx^2}+k_0\frac{dG}{dx}+(-l(l+1)+k_l)G+\frac{1}{x}k_1\frac{d}{dx}G=0 \tag{9}$$

We look for an homogeneous solution $H_1(x)$ by making a series expansion

$$H_1(x) = a_0 + a_2x^2 + a_3x^3 + ...$$

. The coefficients a_0 and a_2 must satisfy the relation

$$2(1+k_1)a_2 + (-l(l+1)+k_l)a_0 = 0 \tag{10}$$

we see that $k_1 = -1$ Tthe pathological case $a_0 = 0$ and the nonvanishing homogeneous solution does not exist.

A second homogeneous solution $H_2(x)$ can be found by searching a solution that vanishes at $x = 0$, $(r = 1)$. We put $H_2(x) = x^j$ we obtain

$$j(j-1) + jk_1 = 0$$

from which

$$j = -k_1 + 1 \tag{11}$$

Therefore $H_2(x)$ will be

$$H_2(x) = x^j(1 + a_{j+1}x + ...) \tag{12}$$

where j is given by the Eq.(11) we see that if $k_1 < 1$ then the solution is regular, moreover if k_1 is integer number $k_1 \leq -2$ the solution has a polynomial behavior near the singularity.
Conclusions: If $k_1 < 2$ Then it exist two independent homogeneous solutions of the equation Eq.8

Numerical solution of the homogeneous equations

If a non vanishing solution exists we shall proceed in the following way take a solution of the first order differential equation appearing in the singular term of the Eq.(8):

$$g_0 = r^{\frac{-k_2}{k_1}} \tag{13}$$

This solution, in general is not a solution of the the second order equation (8) Introduce g_0 in the Eq.(8) and compute the rest R. Solve the non homogeneous equation

$$\frac{d^2G}{dr^2}+\frac{k_0}{r}\frac{dG}{dr}+\frac{1}{r^2}(k_l-l(l+1))G+\frac{1}{x}(k_1\frac{d}{dr}+\frac{k_2}{r})G = -R \tag{14}$$

with the Galerkin approximation by using a new set of function Φ_n vanishing as x^2. We can use the set of (non orthogonal functions)

$$\Phi_n = (r-1)^2 T_n(r) \tag{15}$$

Let be g_p this particular solution, The homogeneous H_1 solution of the EQ.(8) will be

$$H_1 = g_p + g_0 \tag{16}$$

Numerical implementation

In this section I will show how to find numerically the homogeneouwe see that if $k_1 < 1$ then the solution is regular, moreover if k_1 is integer number $k_1 \leq -2$ the solution has a polynomial behavior near the singularity.
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$$H_1=g_p+g_0 \tag{20}$$

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In this section I will show how to find numerically the hogeneos solutions. We shall consider the solution H_2 that vanishes at $r = 1$

Let be $\mathcal{O}_i^j$ the matrix of the operator of the equation Eq.(9)

$$\mathcal{O} = r^2 \frac{d^2}{dr^2} + r\frac{d}{dr} + k_l - l(l+1) + \frac{r}{x}(rk_1\frac{d}{dr} + k_2) \tag{21}$$

with respect the Galerkin basis

$$\Phi_n(r) = (r-1)^2 T_n(r)$$

Finding H_2 it means to find the coefficients a_n of the expansion

$$H_2(r) = \sum a_n \Phi_n(r)$$

Consequently we have to find a non trivial solution of the algebraic system of equations

$$\mathcal{O}_i^j\, a_j = 0 \tag{22}$$

A such a solution exists because the determinant of the matrix $\mathcal{O}_i^j$ vanishes. We shall replace the last line of the system (22) by

$$\mathcal{O}_N^j = 1, 0, 0, ...$$

and we impose that the first coefficient a_1=1 the system will look as

$$\mathcal{O}_1^1\, a_1 + \mathcal{O}_1^2\, a_2 + \mathcal{O}_1^3\, a_3 + ... = 0$$

$$\mathcal{O}_2^1\, a_1 + \mathcal{O}_2^2\, a_2 + \mathcal{O}_2^3\, a_3 + ... = 0$$

$$.................................... = 0$$

$$a_1 + 0 + 0 + 0 + 0 + 0 + ... = 1$$

Solution of the inhomogeneos equations
(The pathological case)

We shall consider the solution of the thoroidal component of the shift:

$$r^2\frac{d^2\mu}{dr^2}+2r\frac{d\mu}{dr}-l(l+1)\mu+\frac{r^2}{x}(-\frac{d\mu}{dr}+\frac{\mu}{r}+S_1) = r^2 S_2 \tag{23}$$

$(x = r - 1)$ The case $k_1 = -1$ is pathological, because it exists a non vanishing homogeneous soluttion at $r = 1$ only for $l = 1$. In order to handle the singular term S_1/x we define a new function

$$\tilde{S}_1 = S_1(r) - q$$

where $q = S_1(1)$. Thus function vanishes at $r = 1$ an we re=writw the above equation as

$$r^2\frac{d^2\mu}{dr^2}+2r\frac{d\mu}{dr}-l(l+1)\mu+\frac{r^2}{x}(-\frac{d\mu}{dr}+\frac{\mu}{r}+q) = r^2(S_2+\frac{\tilde{S}_1}{x}) \quad (24)$$

We look for a solution $\tilde{\mu}$ such thta

$$\tilde{\mu} = -qx + F(r)$$

where Fr vanishes as x^2 at $r = 1$. By replacing $\tilde{\mu}$ we have

$$r^2\frac{d^2F}{dr^2}+2r\frac{dF}{dr}-l(l+1)F+\frac{r^2}{x}(-\frac{dF}{dr}+\frac{F}{r}) \quad (25)$$

$$= r^2[S_2 + \frac{\tilde{S}_1}{x} - q(3r - l(l+1)x] \quad (26)$$

and the solution is obtained by expanding F on the Galerkin base as was done beore.

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[2]Note that if a regular solution is required, the source must vanishes at$r = 1$

Finding Kerr solution startinf from nothing

We show how construct an apprxomitatd Kerr solution as an application of the above formalism.

Start frm the flat metric f_{ik}.
First step:
Find a solution of the lapse equation

$$\triangle N = 0$$

with the B,C. $N(1) = 0$, $N(\infty) = 1$ This solution can be

$$N(r) = 1 - \frac{1}{r}$$

Second step:
Find a solution of the linearised eequation for Ψ^4

$$\triangle \Psi^4 = 0$$

where Ψ is the conformal factor. The solution must satisfy the B.C. of an apparent horizon

$$\frac{d\Psi^4}{dr} = -1 \,|_{r=1}, \quad \Psi^4 = 1 \,|_{r=\infty}$$

This solution is

$$\Psi^4 = \frac{1}{r}$$

3^{th} step: Find (numerically) a solution for μ_l with the B.C.

$$\mu_l = \delta_l^1 \mu_0 \,|_{r=1}, \qquad \mu_l = 0 \,|_{r=\infty}$$

where δ_l^i is the Kronler δ Iterate
Note that th source of μ vanishes at $r = 1$ at each iterartion. (See footnote)

Fig. 1) shows the lapse N. Fig.2) shows the fonction $N_0 = N/x$ (in the first domain) Fig. 3) the shift (for different values of θ
The other figures show the convergence of the iteration.

Conclusion

We have studied the analytical properties of the

solutions of singular elliptical P.D.E. Spectral methods allows to us to compute numerical solutions of singular equations.
As examplese computed the Kerr solution within the conformally flat approximation. The algorithm has shown to be robust (in the sense that it converges exponentially without a relaxation parameter).

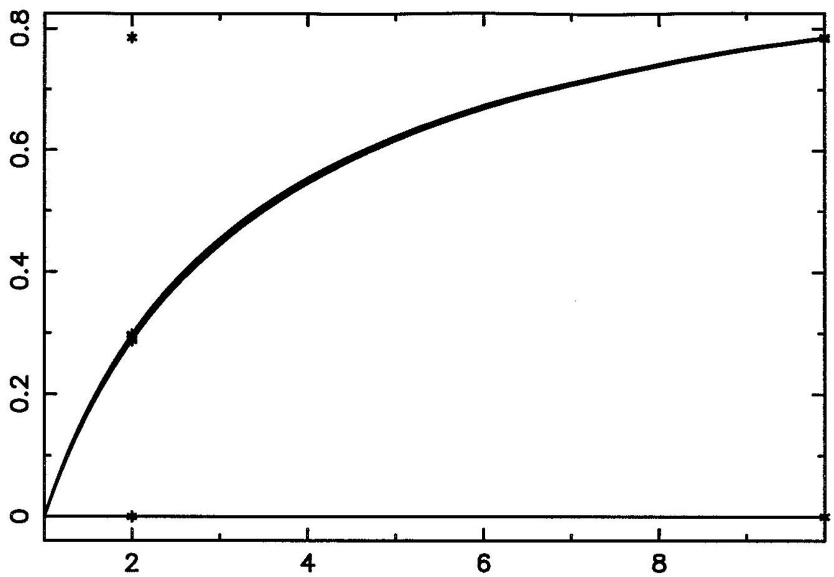

Fig. 1

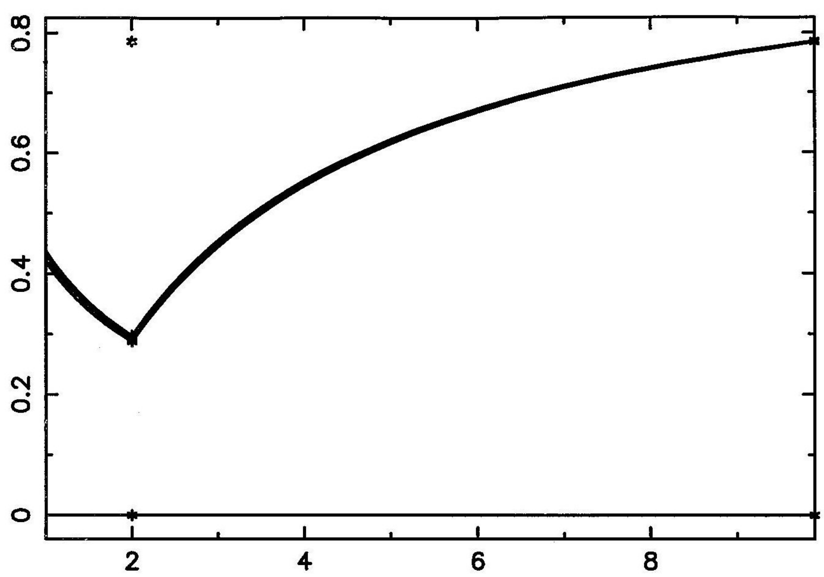

Fig. 2

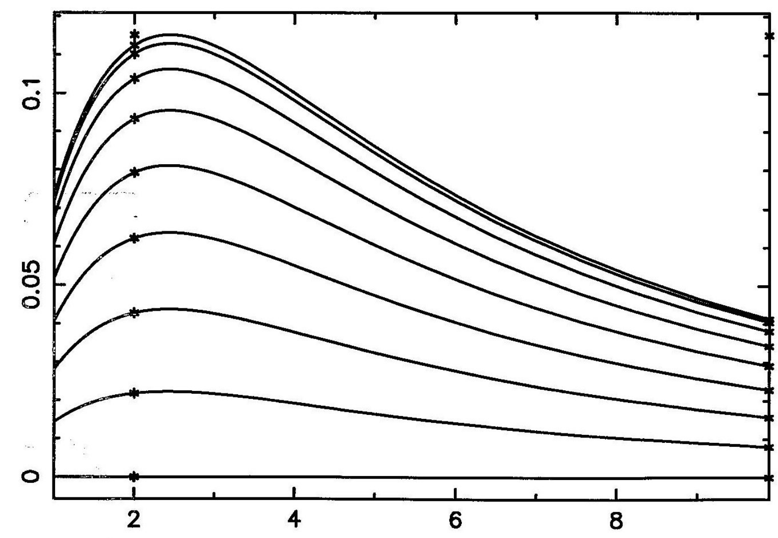

Fig. 3

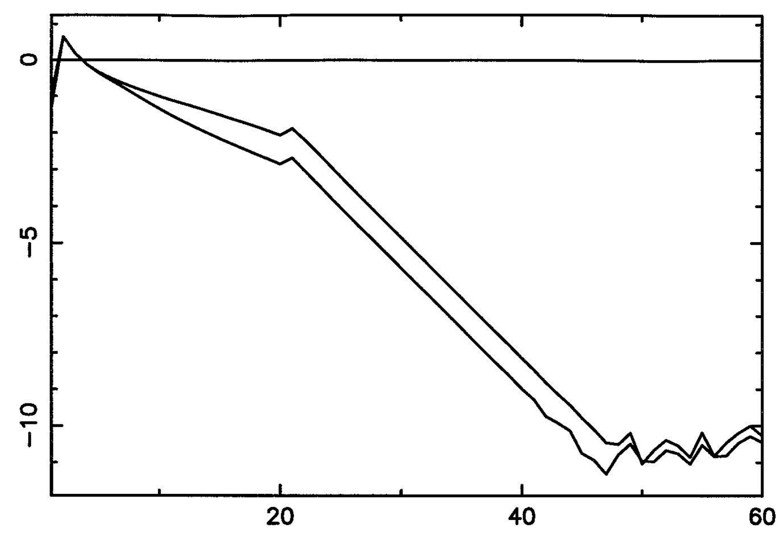

Fig. 4

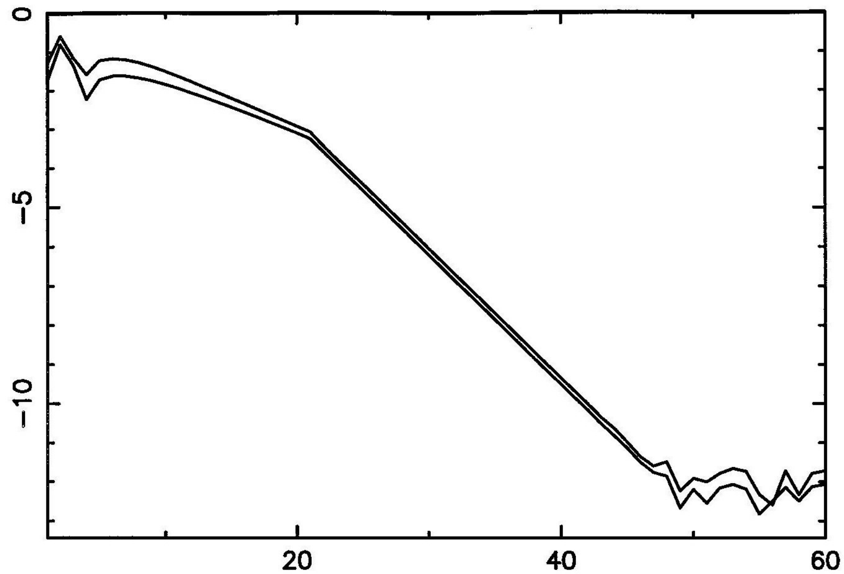

Fig. 5

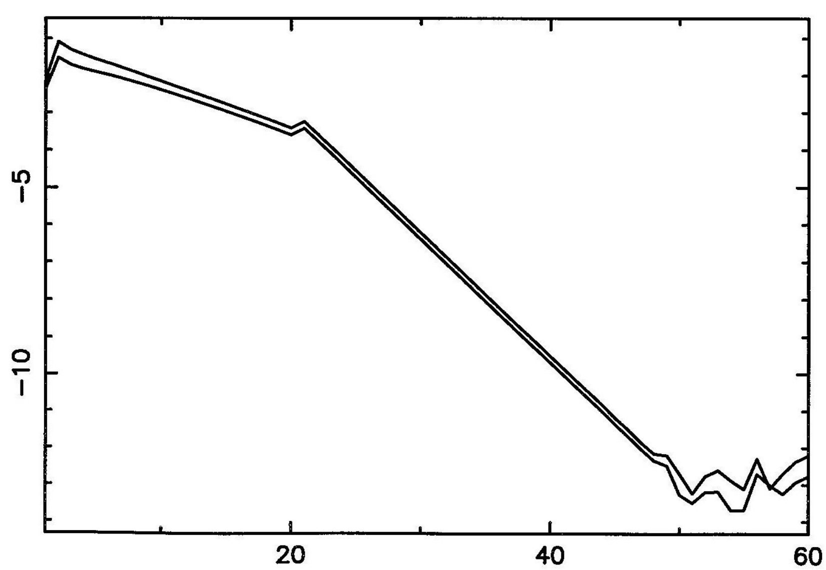

Fig. 6