An introduction to polynomial interpolation

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Outline

1. Introduction
2. Interpolation on an arbitrary grid
3. Expansions onto orthogonal polynomials
4. Convergence of the spectral expansions
5. References
Basic idea: approximate functions $\mathbb{R} \rightarrow \mathbb{R}$ by polynomials

Polynomials are the only functions that a computer can evaluate exactly.

Two types of numerical methods based on polynomial approximations:

- **spectral methods**: high order polynomials on a single domain (or a few domains)
- **finite elements**: low order polynomials on many domains
Basic idea: approximate functions $\mathbb{R} \to \mathbb{R}$ by **polynomials**

Polynomials are the only functions that a computer can evaluate exactly.

Two types of numerical methods based on polynomial approximations:

- **spectral methods**: high order polynomials on a single domain (or a few domains)
- **finite elements**: low order polynomials on many domains
We consider real-valued functions on the compact interval $[-1, 1]$:

$$f : [-1, 1] \rightarrow \mathbb{R}$$

We denote

- by $\mathbb{P}$ the set all real-valued polynomials on $[-1, 1]$:

$$\forall p \in \mathbb{P}, \forall x \in [-1, 1], \ p(x) = \sum_{i=0}^{n} a_i x^i$$

- by $\mathbb{P}_N$ (where $N$ is a positive integer), the subset of polynomials of degree at most $N$. 
Is it a good idea to approximate functions by polynomials?

For **continuous functions**, the answer is **yes**:

**Theorem (Weierstrass, 1885)**

\[ \mathbb{P} \text{ is a dense subspace of the space } C^0([-1, 1]) \text{ of all continuous functions on } [-1, 1], \text{ equiped with the uniform norm } \| \cdot \|_\infty. \]

\(^a\)This is a particular case of the **Stone-Weierstrass theorem**

The **uniform norm** or **maximum norm** is defined by \[ \|f\|_\infty = \max_{x \in [-1, 1]} |f(x)| \]

Other phrasings:

For any continuous function on \([-1, 1]\), \(f\), and any \(\epsilon > 0\), there exists a polynomial \(p \in \mathbb{P}\) such that \[ \|f - p\|_\infty < \epsilon. \]

For any continuous function on \([-1, 1]\), \(f\), there exists a sequence of polynomials \((p_n)_{n \in \mathbb{N}}\) which converges uniformly towards \(f\): \[ \lim_{n \to \infty} \|f - p_n\|_\infty = 0. \]
For a given continuous function: $f \in C^0([−1, 1])$, a best approximation polynomial of degree $N$ is a polynomial $p^*_N(f) \in \mathbb{P}_N$ such that

$$\|f - p^*_N(f)\|_\infty = \min \{\|f - p\|_\infty, \ p \in \mathbb{P}_N\}$$

**Chebyshev’s alternant theorem (or equioscillation theorem)**

For any $f \in C^0([−1, 1])$ and $N \geq 0$, the best approximation polynomial $p^*_N(f)$ exists and is unique. Moreover, there exists $N + 2$ points $x_0, x_1, \ldots, x_{N+1}$ in $[-1,1]$ such that

$$f(x_i) - p^*_N(f)(x_i) = (-1)^i \|f - p^*_N(f)\|_\infty, \quad 0 \leq i \leq N + 1$$

or

$$f(x_i) - p^*_N(f)(x_i) = (-1)^{i+1} \|f - p^*_N(f)\|_\infty, \quad 0 \leq i \leq N + 1$$

**Corollary**: $p^*_N(f)$ interpolates $f$ in $N + 1$ points.
Illustration of Chebyshev’s alternant theorem

\[ N = 1 \]

\[ \| f - p_1^*(f) \|_\infty = \| f - p_1^*(f) \|_\infty \]

Graph showing polynomial approximation with Chebyshev's alternant theorem.
Illustration of Chebyshev’s alternant theorem

\[ N = 1 \]

\[ \| f - p_1^*(f) \|_\infty = \sum_{i=0}^{N-1} \frac{|f(x_i) - p_1^*(x_i)|}{x_{i+1} - x_i} \]

\( x_0, x_1, x_2 \)
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Interpolation on an arbitrary grid

Definition: given an integer \( N \geq 1 \), a grid is a set of \( N + 1 \) points \( X = (x_i)_{0 \leq i \leq N} \) in \([-1,1]\) such that \(-1 \leq x_0 < x_1 < \cdots < x_N \leq 1\). The \( N + 1 \) points \((x_i)_{0 \leq i \leq N}\) are called the nodes of the grid.

Theorem

Given a function \( f \in C^0([-1,1]) \) and a grid of \( N + 1 \) nodes, \( X = (x_i)_{0 \leq i \leq N} \), there exist a unique polynomial of degree \( N \), \( I_X^N f \), such that

\[
I_X^N f(x_i) = f(x_i), \quad 0 \leq i \leq N
\]

\( I_X^N f \) is called the interpolant (or the interpolating polynomial) of \( f \) through the grid \( X \).
The interpolant $I^X_N f$ can be expressed in the Lagrange form:

$$I^X_N f(x) = \sum_{i=0}^{N} f(x_i) \ell^X_i (x),$$

where $\ell^X_i (x)$ is the $i$-th Lagrange cardinal polynomial associated with the grid $X$:

$$\ell^X_i (x) := \prod_{\substack{j=0 \atop j \neq i}}^{N} \frac{x - x_j}{x_i - x_j}, \quad 0 \leq i \leq N$$

The Lagrange cardinal polynomials are such that

$$\ell^X_i (x_j) = \delta_{ij}, \quad 0 \leq i, j \leq N$$
Interpolation on an arbitrary grid

Examples of Lagrange polynomials

Uniform grid $N = 8$  $\ell^X_0(x)$

Lagrange polynomials
Uniform grid $N = 8$  $\ell^X_1(x)$

Lagrange polynomials
Examples of Lagrange polynomials

Uniform grid $N = 8$ \( \ell_2^X(x) \)

Lagrange polynomials
Examples of Lagrange polynomials

Uniform grid $N = 8$  \( \ell_3^X(x) \)
Examples of Lagrange polynomials

Uniform grid $N = 8$ \quad $\ell_4^X(x)$

Lagrange polynomials
Examples of Lagrange polynomials

Uniform grid $N = 8$ \[ \ell_5^X(x) \]

Lagrange polynomials
Examples of Lagrange polynomials

Uniform grid $N = 8$ \quad $\ell_6^X(x)$
Examples of Lagrange polynomials

Uniform grid $N = 8$ \ $\ell^X_7(x)$
Examples of Lagrange polynomials

Uniform grid $N = 8$ \(-\ell_8^X(x)\)

Lagrange polynomials
Examples of Lagrange polynomials

Uniform grid $N = 8$

Lagrange polynomials
Interpolation error with respect to the best approximation error

Let \( N \in \mathbb{N} \), \( X = (x_i)_{0 \leq i \leq N} \) a grid of \( N + 1 \) nodes and \( f \in C^0([-1, 1]) \).

Let us consider the interpolant \( I^X_N f \) of \( f \) through the grid \( X \).

The best approximation polynomial \( p^*_N(f) \) is also an interpolant of \( f \) at \( N + 1 \) nodes (in general different from \( X \)).

How does the error \( \|f - I^X_N f\|_\infty \) behave with respect to the smallest possible error \( \|f - p^*_N(f)\|_\infty \)?

The answer is given by the formula:

\[
\|f - I^X_N f\|_\infty \leq (1 + \Lambda_N(X)) \|f - p^*_N(f)\|_\infty
\]

where \( \Lambda_N(X) \) is the Lebesgue constant relative to the grid \( X \):

\[
\Lambda_N(X) := \max_{x \in [-1,1]} \sum_{i=0}^{N} |\ell^X_i(x)|
\]
The Lebesgue constant contains all the information on the effects of the choice of \( X \) on \( \| f - I_X^N f \|_\infty \).

**Theorem (Erdős, 1961)**

For any choice of the grid \( X \), there exists a constant \( C > 0 \) such that

\[
\Lambda_N(X) > \frac{2}{\pi} \ln(N + 1) - C
\]

**Corollary:** \( \Lambda_N(X) \to \infty \) as \( N \to \infty \)

In particular, for a uniform grid, \( \Lambda_N(X) \sim \frac{2^{N+1}}{eN \ln N} \) as \( N \to \infty \)!

This means that for any choice of type of sampling of \([-1, 1]\), there exists a continuous function \( f \in C^0([-1, 1]) \) such that \( I_X^N f \) does not convergence uniformly towards \( f \).
Example: uniform interpolation of a “gentle” function

\[ f(x) = \cos(2 \exp(x)) \] uniform grid \( N = 4 \):

\[ \| f - I^X_4 f \|_\infty \approx 1.40 \]
Example: uniform interpolation of a "gentle" function

\[ f(x) = \cos(2 \exp(x)) \text{ uniform grid } N = 6 : \|f - I^X_6 f\|_\infty \simeq 1.05 \]
Interpolation on an arbitrary grid

Example: uniform interpolation of a "gentle" function

\[ f(x) = \cos(2 \exp(x)) \] uniform grid \( N = 8 \): \( \| f - I^X_{8} f \|_{\infty} \approx 0.13 \)
Example: uniform interpolation of a "gentle" function

\[ f(x) = \cos(2 \exp(x)) \text{ uniform grid } N = 12 : \|f - I_{12}^X f\|_\infty \approx 0.13 \]
Interpolation on an arbitrary grid

Example: uniform interpolation of a “gentle” function

\[ f(x) = \cos(2 \exp(x)) \] uniform grid \( N = 16 \):

\[ \| f - I_{16}^X f \|_\infty \approx 0.025 \]
Example: uniform interpolation of a “gentle” function

\[ f(x) = \cos(2 \exp(x)) \] uniform grid \( N = 24 \): \[ \|f - I^X_{24}f\|_\infty \approx 4.6 \times 10^{-4} \]
Interpolation on an arbitrary grid

Runge phenomenon

\[ f(x) = \frac{1}{1 + 16x^2} \quad \text{uniform grid } N = 4 : \quad \| f - I^X f \|_\infty \simeq 0.39 \]
$f(x) = \frac{1}{1 + 16x^2}$  

uniform grid $N = 6$ : $\|f - I_6^X f\|_\infty \simeq 0.49$
Runge phenomenon

\[ f(x) = \frac{1}{1 + 16x^2} \quad \text{uniform grid } N = 8 : \| f - I_8^x f \|_\infty \simeq 0.73 \]
$f(x) = \frac{1}{1 + 16x^2}$

uniform grid $N = 12$:

$\|f - I_{12}^X f\|_\infty \approx 1.97$
Runge phenomenon

\[ f(x) = \frac{1}{1 + 16x^2} \quad \text{uniform grid } N = 16 : \| f - I_{16}^X f \|_\infty \simeq 5.9 \]
Runge phenomenon

\[ f(x) = \frac{1}{1 + 16x^2} \]  
uniform grid \( N = 24 \): \( \| f - I_{24}^X f \|_{\infty} \approx 62 \)
Let us assume that the function $f$ is sufficiently smooth to have derivatives at least up to the order $N + 1$, with $f^{(N+1)}$ continuous, i.e. $f \in C^{N+1}([-1, 1])$.

**Theorem (Cauchy)**

If $f \in C^{N+1}([-1, 1])$, then for any grid $X$ of $N + 1$ nodes, and for any $x \in [-1, 1]$, the interpolation error at $x$ is

$$f(x) - I^X_N(x) = \frac{f^{(N+1)}(\xi)}{(N + 1)!} \omega^X_{N+1}(x)$$

where $\xi = \xi(x) \in [-1, 1]$ and $\omega^X_{N+1}(x)$ is the nodal polynomial associated with the grid $X$.

**Definition:** The nodal polynomial associated with the grid $X$ is the unique polynomial of degree $N + 1$ and leading coefficient $1$ whose zeros are the $N + 1$ nodes of $X$:

$$\omega^X_{N+1}(x) := \prod_{i=0}^{N} (x - x_i)$$
Example of nodal polynomial

Uniform grid $N = 8$

Nodal polynomial
Minimizing the interpolation error by the choice of grid

In Eq. (1), we have no control on \( f^{(N+1)} \), which can be large.
For example, for \( f(x) = 1/(1 + \alpha^2 x^2) \), \( \|f^{(N+1)}\|_\infty = (N + 1)! \alpha^{N+1} \).

Idea: choose the grid \( X \) so that \( \omega_{N+1}^X(x) \) is small, i.e. \( \|\omega_{N+1}^X\|_\infty \) is small.

Notice: \( \omega_{N+1}^X(x) \) has leading coefficient 1: \( \omega_{N+1}^X(x) = x^{N+1} + \sum_{i=0}^{N} a_i x^i \).

Theorem (Chebyshev)

Among all the polynomials of degree \( N + 1 \) and leading coefficient 1, the unique polynomial which has the smallest uniform norm on \([-1, 1]\) is the \((N + 1)\)-th Chebyshev polynomial divided by \(2^N\): \( T_{N+1}(x)/2^N \).

Since \( \|T_{N+1}\|_\infty = 1 \), we conclude that if we choose the grid nodes \((x_i)_{0 \leq i \leq N}\) to be the \( N + 1 \) zeros of the Chebyshev polynomial \( T_{N+1} \), we have

\[
\|\omega_{N+1}^X\|_\infty = \frac{1}{2^N}
\]

and this is the smallest possible value.
Chebyshev-Gauss grid

The grid $X = (x_i)_{0 \leq i \leq N}$ such that the $x_i$’s are the $N + 1$ zeros of the Chebyshev polynomial of degree $N + 1$ is called the Chebyshev-Gauss (CG) grid. It has much better interpolation properties than the uniform grid considered so far. In particular, from Eq. (1), for any function $f \in C^{N+1}([-1, 1])$,

$$\|f - I_N^{CG} f\|_{\infty} \leq \frac{1}{2^N (N + 1)!} \|f^{(N+1)}\|_{\infty}$$

If $f^{(N+1)}$ is uniformly bounded, the convergence of the interpolant $I_N^{CG} f$ towards $f$ when $N \to \infty$ is then extremely fast.

Also the Lebesgue constant associated with the Chebyshev-Gauss grid is small:

$$\Lambda_N(CG) \sim \frac{2}{\pi} \ln(N + 1) \quad \text{as} \quad N \to \infty$$

This is much better than uniform grids and close to the optimal value.
Example: Chebyshev-Gauss interpolation of \( f(x) = \frac{1}{1+16x^2} \)

CG grid \( N = 4 \): \( \| f - I_{4}^{CG} f \|_{\infty} \approx 0.31 \)
Example: Chebyshev-Gauss interpolation of \( f(x) = \frac{1}{1 + 16x^2} \)

CG grid \( N = 6 \): \( \| f - I_{6}^{CG} f \|_{\infty} \approx 0.18 \)
Example: Chebyshev-Gauss interpolation of $f(x) = \frac{1}{1+16x^2}$

$\|f - I_{8}^{\text{CG}} f\|_{\infty} \simeq 0.10$
Example: Chebyshev-Gauss interpolation of 

\[ f(x) = \frac{1}{1 + 16x^2} \]

CG grid \( N = 12 \): \( \| f - I_{12}^{CG} f \|_\infty \simeq 3.8 \times 10^{-2} \)
Example: Chebyshev-Gauss interpolation of \( f(x) = \frac{1}{1+16x^2} \)

CG grid \( N = 16 \): \( \| f - I_{16}^{CG} f \|_\infty \simeq 1.5 \times 10^{-2} \)
Example: Chebyshev-Gauss interpolation of $f(x) = \frac{1}{1 + 16x^2}$

$$f(x) = \frac{1}{1 + 16x^2}$$

CG grid $N = 24$: $\|f - I_{24}^{CG}f\|_\infty \simeq 2.0 \times 10^{-3}$

no Runge phenomenon!
Example: Chebyshev-Gauss interpolation of \( f(x) = \frac{1}{1+16x^2} \)

Variation of the interpolation error as \( N \) increases
The Chebyshev polynomials, the zeros of which provide the Chebyshev-Gauss nodes, constitute a family of orthogonal polynomials, and the Chebyshev-Gauss nodes are associated to Gauss quadratures.
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Hilbert space $L^2_w(-1, 1)$

Framework: Let us consider the functional space

$$L^2_w(-1, 1) = \left\{ f : (-1, 1) \to \mathbb{R}, \quad \int_{-1}^{1} f(x)^2 w(x) \, dx < \infty \right\}$$

where $w : (-1, 1) \to (0, \infty)$ is an integrable function, called the weight function.

$L^2_w(-1, 1)$ is a Hilbert space for the scalar product

$$(f | g)_w := \int_{-1}^{1} f(x) g(x) w(x) \, dx$$

with the associated norm

$$\|f\|_w := (f | f)_w^{1/2}$$
Orthogonal polynomials

The set $\mathbb{P}$ of polynomials on $[-1, 1]$ is a subspace of $L^2_w(-1, 1)$. A family of orthogonal polynomials is a set $(p_i)_{i \in \mathbb{N}}$ such that

- $p_i \in \mathbb{P}$
- $\deg p_i = i$
- $i \neq j \Rightarrow (p_i|p_j)_w = 0$

$(p_i)_{i \in \mathbb{N}}$ is then a basis of the vector space $\mathbb{P}$: $\mathbb{P} = \text{span} \{p_i, i \in \mathbb{N}\}$

**Theorem**

A family of orthogonal polynomial $(p_i)_{i \in \mathbb{N}}$ is a Hilbert basis of $L^2_w(-1, 1)$:

$$\forall f \in L^2_w(-1, 1), \quad f = \sum_{i=0}^{\infty} \tilde{f}_i p_i$$

with $\tilde{f}_i := \frac{(f|p_i)_w}{\|p_i\|^2_w}$.

The above infinite sum means

$$\lim_{N \to \infty} \left\| f - \sum_{i=0}^{N} \tilde{f}_i p_i \right\|_w = 0$$
Jacobi polynomials

Jacobi polynomials are orthogonal polynomials with respect to the weight

\[ w(x) = (1 - x)^\alpha (1 + x)^\beta \]

Subcases:
- Legendre polynomials \( P_n(x) \): \( \alpha = \beta = 0 \), i.e. \( w(x) = 1 \)
- Chebyshev polynomials \( T_n(x) \): \( \alpha = \beta = -\frac{1}{2} \), i.e. \( w(x) = \frac{1}{\sqrt{1 - x^2}} \)

Jacobi polynomials are eigenfunctions of the singular\(^1\) Sturm-Liouville problem

\[ -\frac{d}{dx} \left[ (1 - x^2) w(x) \frac{du}{dx} \right] = \lambda w(x) u, \quad x \in (-1, 1) \]

\(^1\text{singular}\) means that the coefficient in front of \( du/dx \) vanishes at the extremities of the interval \([-1, 1]\)
Legendre polynomials

\[ w(x) = 1: \quad \int_{-1}^{1} P_i(x) P_j(x) \, dx = \frac{2}{2i + 1} \delta_{ij} \]

\( P_0(x) = 1 \)
\( P_1(x) = x \)
\( P_2(x) = \frac{1}{2} (3x^2 - 1) \)
\( P_3(x) = \frac{1}{2} (5x^3 - 3x) \)
\( P_4(x) = \frac{1}{8} (35x^4 - 30x^2 + 3) \)

\( P_{i+1}(x) = \frac{2i+1}{i+1} x P_i(x) - \frac{i}{i+1} P_{i-1}(x) \)
Expansions onto orthogonal polynomials

Chebyshev polynomials

\[ w(x) = \frac{1}{\sqrt{1 - x^2}} : \int_{-1}^{1} T_i(x)T_j(x) \frac{dx}{\sqrt{1 - x^2}} = \frac{\pi}{2} (1 + \delta_{0i}) \delta_{ij} \]

\[ T_0(x) = 1 \]
\[ T_1(x) = x \]
\[ T_2(x) = 2x^2 - 1 \]
\[ T_3(x) = 4x^3 - 3x \]
\[ T_4(x) = 8x^4 - 8x^2 + 1 \]

\[ \cos(n\theta) = T_n(\cos \theta) \]

\[ T_{i+1}(x) = 2xT_i(x) - T_{i-1}(x) \]

Chebyshev polynomials up to N=8
Expansions onto orthogonal polynomials

Legendre and Chebyshev compared

[from Fornberg (1998)]
Let us consider $f \in L^2_w(-1,1)$ and a family $(p_i)_{i \in \mathbb{N}}$ of orthogonal polynomials with respect to the weight $w$. Since $(p_i)_{i \in \mathbb{N}}$ is a Hilbert basis of $L^2_w(-1,1)$ we have

$$f(x) = \sum_{i=0}^{\infty} \tilde{f}_i p_i(x)$$

with $\tilde{f}_i := \frac{(f|p_i)_w}{\|p_i\|_w^2}$.

The truncated sum

$$\Pi^w_N f(x) := \sum_{i=0}^{N} \tilde{f}_i p_i(x)$$

is a polynomial of degree $N$: it is the orthogonal projection of $f$ onto the finite dimensional subspace $\mathbb{P}_N$ with respect to the scalar product $(\cdot|\cdot)_w$. We have

$$\lim_{N \to \infty} \|f - \Pi^w_N f\|_w = 0$$

Hence $\Pi^w_N f$ can be considered as a polynomial approximation of the function $f$. 
Expansions onto orthogonal polynomials

Example: Chebyshev projection of $f(x) = \cos(2 \exp(x))$

\[ f(x) = \cos(2 \exp(x)) \quad w(x) = (1 - x^2)^{-1/2} \quad N = 4 : \| f - \Pi_4^w f \|_\infty \approx 0.66 \]
Example: Chebyshev projection of $f(x) = \cos(2 \exp(x))$

$f(x) = \cos(2 \exp(x)) \quad w(x) = (1 - x^2)^{-1/2} \quad N = 6 : \|f - \Pi_6^w f\|_\infty \approx 0.30$
Example: Chebyshev projection of $f(x) = \cos(2 \exp(x))$

$f(x) = \cos(2 \exp(x))$  \quad  w(x) = (1 - x^2)^{-1/2}  \quad  N = 8 : \|f - \Pi_8 w f\|_\infty \simeq 4.9 \times 10^{-2}$
Example: Chebyshev projection of $f(x) = \cos(2 \exp(x))$

$$f(x) = \cos(2 \exp(x)) \quad w(x) = (1 - x^2)^{-1/2} \quad N = 12 : \|f - \Pi_{12}^w f\|_\infty \approx 6.1 \, 10^{-3}$$
Example: Chebyshev projection of \( f(x) = \cos(2 \exp(x)) \)

Variation of the projection error \( \| f - \Pi_N^w f \|_\infty \) as \( N \) increases.
The coefficients $\tilde{f}_i$ of the orthogonal projection of $f$ are given by

$$\tilde{f}_i := \frac{(f|p_i)_w}{\|p_i\|_w^2} = \frac{1}{\|p_i\|_w^2} \int_{-1}^{1} f(x) p_i(x) w(x) \, dx$$

(2)

**Problem:** the above integral cannot be computed exactly; we must seek a numerical approximation.

**Solution:** Gaussian quadrature
Theorem (Gauss, Jacobi)

Let \((p_i)_{i \in \mathbb{N}}\) be a family of orthogonal polynomials with respect to some weight \(w\). For \(N > 0\), let \(X = (x_i)_{0 \leq i \leq N}\) be the grid formed by the \(N + 1\) zeros of the polynomial \(p_{N+1}\) and

\[
 w_i := \int_{-1}^{1} \ell^X_i(x) w(x) \, dx
\]

where \(\ell^X_i\) is the \(i\)-th Lagrange cardinal polynomial of the grid \(X\). Then

\[
 \forall f \in \mathbb{P}_{2N+1}, \quad \int_{-1}^{1} f(x) w(x) \, dx = \sum_{i=0}^{N} w_i f(x_i)
\]

If \(f \notin \mathbb{P}_{2N+1}\), the above formula provides a good approximation of the integral.
The nodes of the Gauss quadrature, being the zeros of $p_{N+1}$, do not encompass the boundaries $-1$ and $1$ of the interval $[-1, 1]$. For numerical purpose, it is desirable to include these points in the boundaries.

This possible at the price of reducing by 2 units the degree of exactness of the Gauss quadrature.
Theorem (Gauss-Lobatto quadrature)

Let \((p_i)_{i \in \mathbb{N}}\) be a family of orthogonal polynomials with respect to some weight \(w\). For \(N > 0\), let \(X = (x_i)_{0 \leq i \leq N}\) be the grid formed by the \(N + 1\) zeros of the polynomial

\[ q_{N+1} = p_{N+1} + \alpha p_N + \beta p_{N-1} \]

where the coefficients \(\alpha\) and \(\beta\) are such that \(x_0 = -1\) and \(x_N = 1\).

Let

\[ w_i := \int_{-1}^{1} \ell_i^X(x) w(x) \, dx \]

where \(\ell_i^X\) is the \(i\)-th Lagrange cardinal polynomial of the grid \(X\).

Then

\[ \forall f \in \mathbb{P}_{2N-1}, \quad \int_{-1}^{1} f(x) w(x) \, dx = \sum_{i=0}^{N} w_i f(x_i) \]

Notice: \(f \in \mathbb{P}_{2N-1}\) instead of \(f \in \mathbb{P}_{2N+1}\) for Gauss quadrature.
Remark: if the \((p_i)\) are Jacobi polynomials, i.e. if \(w(x) = (1 - x)^\alpha(1 + x)^\beta\), then the Gauss-Lobatto nodes which are strictly inside \((-1, 1)\), i.e. \(x_1, \ldots, x_{N-1}\), are the \(N - 1\) zeros of the polynomial \(p'_N\), or equivalently the points where the polynomial \(p_N\) is extremal.

This of course holds for Legendre and Chebyshev polynomials. For Chebyshev polynomials, the Gauss-Lobatto nodes and weights have simple expressions:

\[
x_i = -\cos \frac{\pi i}{N}, \quad 0 \leq i \leq N
\]

\[
w_0 = w_N = \frac{\pi}{2N}, \quad w_i = \frac{\pi}{N}, \quad 1 \leq i \leq N - 1
\]

Note: in the following, we consider only Gauss-Lobatto quadratures
The Gauss-Lobatto quadrature motivates the introduction of the following scalar product:

\[ \langle f | g \rangle_N = \sum_{i=0}^{N} w_i f(x_i) g(x_i) \]

It is called the **discrete scalar product** associated with the Gauss-Lobatto nodes \( X = (x_i)_{0 \leq i \leq N} \).

Setting \( \gamma_i := \langle p_i | p_i \rangle_N \), the **discrete coefficients** associated with a function \( f \) are given by

\[ \hat{f}_i := \frac{1}{\gamma_i} \langle f | p_i \rangle_N, \quad 0 \leq i \leq N \]

which can be seen as approximate values of the coefficients \( \tilde{f}_i \) provided by the Gauss-Lobatto quadrature [cf. Eq. (2)].
Let \( I^\text{GL}_N f \) be the interpolant of \( f \) at the Gauss-Lobatto nodes \( X = (x_i)_{0 \leq i \leq N} \). Being a polynomial of degree \( N \), it is expandable as

\[
I^\text{GL}_N f(x) = \sum_{i=0}^{N} a_i p_i(x)
\]

Then, since \( I^\text{GL}_N f(x_j) = f(x_j) \),

\[
\hat{f}_i = \frac{1}{\gamma_i} \langle f | p_i \rangle_N = \frac{1}{\gamma_i} \langle I^\text{GL}_N f | p_i \rangle_N = \frac{1}{\gamma_i} \sum_{j=0}^{N} a_j \langle p_j | p_i \rangle_N
\]

Now, if \( j = i \), \( \langle p_j | p_i \rangle_N = \gamma_i \) by definition. If \( j \neq i \), \( p_j p_i \in \mathbb{P}_{2N-1} \) so that the Gauss-Lobatto formula holds and gives \( \langle p_j | p_i \rangle_N = (p_j | p_i)_w = 0 \). Thus we conclude that \( \langle p_j | p_i \rangle_N = \gamma_i \delta_{ij} \) so that the above equation yields \( \hat{f}_i = a_i \), i.e. the discrete coefficients are nothing but the coefficients of the expansion of the interpolant at the Gauss-Lobatto nodes.
Spectral representation of a function

In a spectral method, the numerical representation of a function $f$ is through its interpolant at the Gauss-Lobatto nodes:

$$I^\text{GL}_N f(x) = \sum_{i=0}^{N} \hat{f}_i p_i(x)$$

The discrete coefficients $\hat{f}_i$ are computed as

$$\hat{f}_i = \frac{1}{\gamma_i} \sum_{j=0}^{N} w_j f(x_j) p_i(x_j)$$

$I^\text{GL}_N f(x)$ is an approximation of the truncated series $\Pi^w_N f(x) = \sum_{i=0}^{N} \tilde{f}_i p_i(x)$, which is the orthogonal projection of $f$ onto the polynomial space $\mathbb{P}_N$. $\Pi^w_N f$ should be the true spectral representation of $f$, but in general it is not computable exactly.

The difference between $I^\text{GL}_N f$ and $\Pi^w_N f$ is called the aliasing error.
Expansions onto orthogonal polynomials

Example: aliasing error for \( f(x) = \cos(2 \exp(x)) \)

\[
f(x) = \cos(2 \exp(x)) \quad w(x) = (1 - x^2)^{-1/2} \quad N = 4
\]

red: \( f \); blue: \( \Pi_N^w f \); green: \( I_{GL}^N f \)
Expansions onto orthogonal polynomials

Example: aliasing error for $f(x) = \cos(2 \exp(x))$

$$f(x) = \cos(2 \exp(x)) \quad w(x) = (1 - x^2)^{-1/2} \quad N = 6$$

red: $f$; blue: $\Pi^w_N f$; green: $I^\text{GL}_N f$
Expansions onto orthogonal polynomials

Example: aliasing error for \( f(x) = \cos(2 \exp(x)) \)

\[ f(x) = \cos(2 \exp(x)) \quad w(x) = (1 - x^2)^{-1/2} \quad N = 8 \]

red: \( f \); blue: \( \prod_{N} w f \); green: \( I_{N}^{GL} f \)
Example: aliasing error for $f(x) = \cos(2 \exp(x))$

\[ f(x) = \cos(2 \exp(x)) \quad w(x) = (1 - x^2)^{-1/2} \quad N = 12 \]

red: $f$; blue: $\prod_{N}^{w} f$; green: $I_{N}^{GL} f$
Expansions onto orthogonal polynomials

Aliasing error = contamination by high frequencies

Aliasing of a $\sin(x)$ wave by a $\sin(5x)$ wave on a 4-points grid
Outline

1 Introduction
2 Interpolation on an arbitrary grid
3 Expansions onto orthogonal polynomials
4 Convergence of the spectral expansions
5 References
Let us consider a function $f \in C^m([-1, 1])$, with $m \geq 0$.

The **Sobolev norm** of $f$ with respect to some weight function $w$ is

$$
\|f\|_{H^m_w} := \left( \sum_{k=0}^{m} \left\| f^{(k)} \right\|_w^2 \right)^{1/2}
$$
Convergence of the spectral expansions

Convergence rates for $f \in C^m([-1, 1])$

**Chebyshev expansions:**

- **truncation error:**
  \[
  \| f - \Pi_N^w f \|_w \leq \frac{C_1}{N^m} \| f \|_{H^m_w} \quad \text{and} \quad \| f - \Pi_N^w f \|_\infty \leq \frac{C_2(1 + \ln N)}{N^m} \sum_{k=0}^{m} \| f^{(k)} \|_\infty
  \]

- **interpolation error:**
  \[
  \| f - I_N^{GL} f \|_w \leq \frac{C_3}{N^m} \| f \|_{H^m_w} \quad \text{and} \quad \| f - I_N^{GL} f \|_\infty \leq \frac{C_4}{N^{m-1/2}} \| f \|_{H^m_w}
  \]

**Legendre expansions:**

- **truncation error:**
  \[
  \| f - \Pi_N^w f \|_w \leq \frac{C_1}{N^m} \| f \|_{H^m_w} \quad \text{and} \quad \| f - \Pi_N^w f \|_\infty \leq \frac{C_2}{N^{m-1/2}} V(f^{(m)})
  \]

- **interpolation error:**
  \[
  \| f - I_N^{GL} f \|_w \leq \frac{C_3}{N^{m-1/2}} \| f \|_{H^m_w}
  \]
If $f \in C^\infty([-1,1])$, the error of the spectral expansions $\Pi_N^w f$ or $I_N^{GL} f$ decays more rapidly than any power of $N$.

In practice: \textbf{exponential decay} \hspace{1cm} \text{example}

This error is called \textbf{evanescent}.
Convergence of the spectral expansions

For non-smooth functions: Gibbs phenomenon

Extreme case: \( f \) discontinuous
Outline

1. Introduction
2. Interpolation on an arbitrary grid
3. Expansions onto orthogonal polynomials
4. Convergence of the spectral expansions
5. References
References