I. THE PROBLEM

We propose to solve the system for a static spherically symmetric Yang-Mills-Higgs monopole. Using the minimal spherically symmetric Ansatz, the solution is described by two functions : one describing the gauge field W and one the Higgs field H. Those two functions depend only on r and obey a system of two coupled equations :

$$W'' = \frac{W\left(W^2-1\right)}{r^2} + WH^2 \tag{1}$$

$$H'' + \frac{2}{r}H' = 2\frac{W^2 H}{r^2} + \frac{\beta^2}{2}H\left(H^2-1\right) \tag{2}$$

The only parameter of the solution is β constraining the mass of the Higgs field. We will restrict ourselves to the cases $0 < \beta < \infty$.

II. ASYMPTOTIC BEHAVIORS

Near the origin, one has the following behaviors :

$$W = 1 - ar^2 + \mathcal{O}\left(r^4\right) \tag{3}$$

$$H = br + \mathcal{O}\left(r^3\right) \tag{4}$$

At infinity, the fields converge exponentially, i.e. W and $h = H - 1$ go to zero exponentially.

III. SYSTEM IN VARIOUS DOMAINS

- **In the nucleus** : one uses $w = \frac{W-1}{r}$ and H (odd functions near the origin) and rewrite the system as :

$$\Delta_{l=1} w \equiv w'' + \frac{2}{r}w' - 2\frac{w}{r^2} = w^3 + 3\frac{w^2}{r} + (1+rw)\frac{H^2}{r} \tag{5}$$

$$\Delta_{l=1} H \equiv H'' + \frac{2}{r}H' - 2\frac{H}{r^2} = 2H\left(w^2 + 2\frac{w}{r}\right) + \frac{\beta^2}{2}H\left(H^2-1\right). \tag{6}$$

- **In the shells** : one uses W and H but rewrites the system to make the Helmholtz operators appear (optional) :

$$\Delta_{l=0} W - W = W\left(H^2-1\right) + \frac{W\left(W^2-1\right)}{r^2} + 2\frac{W'}{r} \tag{7}$$

$$\Delta_{l=0} H - \beta^2 H = 2\frac{W^2 H}{r^2} + \frac{\beta^2}{2}H\left(H^2-3\right) \tag{8}$$

- **In the external domain** : one works with W and $h = H - 1$ and make Helmholtz operators appear :

$$\Delta_{l=0} W - W = hW(h+2) + \frac{W\left(W^2-1\right)}{r^2} + 2\frac{W'}{r} \tag{9}$$

$$\Delta_{l=0} h - \beta^2 h = 2\frac{W^2(h+1)}{r^2} + \frac{\beta^2}{2}h^2(h+3) \tag{10}$$

IV. SUGGESTED STEPS

- Look at the proposed `Monopole` class that contains W, H, w and h (each of them being a `Scalar`).
- Implement functions that initialize W and H, with the right behaviors and basis. Plot the results.
- Implement functions that go from W to w and from H to h and conversely. Plot the various functions.

- Compute the sources in various domains and plot them.
- Setup the main iteration loop, based on `Param_elliptic` class.
- For various moderate values of β, compute a and b appearing in Eqs. (3) and (4).
- Try to go to high values of β.

V. SOLVING THE SYSTEM ON TWO GRIDS

For high values of β, one can show that H varies on a relative length scale $\propto 1/\beta$ whereas W varies always on length of the order unity. So, for high values of β, those two functions vary on very different length scales, causing the code to crash. To cope with that, one can use two grids :

- one on scales of the order 1, used to solve the equation for W.
- one on scales of the order $1/\beta$, used to solve the equation for H

This can be implemented by describing all the fields (W, w, H and h) on two sets of grids. One can go from one grid to the other by using the `Scalar::import()` function. Be careful : this should only be used with continuous functions, to avoid Gibbs phenomenon. Verify that the use of two grids enables to go to very high values of β.